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Numerical Models of Binary Neutron Star System Mergers. I.: Numerical Methods and Equilibrium Data for Newtonian Models

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ABSTRACT

The numerical modeling of binary neutron star mergers has become a subject of much interest in recent years. While a full and accurate model of this phenomenon would require the evolution of the equations of relativistic hydrodynamics along with the Einstein field equations, a qualitative study of the early stages on inspiral can be accomplished by either Newtonian or post-Newtonian models, which are more tractable. However, even purely Newtonian models present numerical challenges that must be overcome in order to have accurate models of the inspiral. In particular, the simulations must maintain conservation of both energy and momenta, and otherwise exhibit good numerical behavior. A spate of recent papers have detailed the results for Newtonian and post-Newtonian models of neutron star coalescence from a variety of groups who employ very different numerical schemes. These include calculations that have been carried out in both inertial and rotating frames, as well as calculations that employ both equilibrium configurations and spherical stars as initial data. However, scant attention has been given to the issue of the the accuracy of the models and the dependence of the results on the computational frame and the initial data. In this paper we offer a comparison of results from both rotating and non-rotating (inertial) frame calculations. We find that the rotating frame calculations offer significantly improved accuracy as compared with the inertial frame models. Furthermore, we show that inertial frame models exhibit significant and erroneous angular momentum loss during the simulations that leads to an unphysical inspiral of the two neutron stars. We also examine the dependence of the models on initial conditions by considering initial configurations that consist of spherical neutron stars as well as stars that are in equilibrium and which are tidally distorted. We compare our models those of Rasio & Shapiro (1992,1994a) and New & Tohline (1997). Finally, we investigate the use of the isolated star approximation for the construction of initial data.

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1. Introduction

1.1. Scientific motivation

Binary Neutron Star Mergers (NSMs) are unique laboratories for the study of astrophysics. The merger involves many elements of the theory of relativistic astrophysics, gravitational wave astronomy, and nuclear astrophysics. Furthermore, NSMs are thought to be a possible source of detectable gravitational radiation, the r-process elements, and gamma ray bursts. In order to develop accurate models of NSMs one must numerically solve the equations describing gas dynamics and the gravitational field arising from the matter. However, the need for accurate numerical models of the inspiraling binary system presents some unique challenges that we address in this paper.

In realistic astrophysical situations the merger of binary neutron star systems is driven by gravitational radiation losses (Misner et al. 1973) . This loss of energy will lead to the inspiral and eventual coalescence of the binary system. The prediction of energy loss by gravitational radiation was confirmed by the observation of PSR1913+16, a binary neutron star system (Hulse & Taylor 1975). The observed rate of decrease of the orbital period of this system is in good agreement with predictions made by general relativity (Taylor & Weisberg 1989). Coalescing binary systems are expected to emit tremendous amounts of energy in the form of gravitational waves during the final stages of coalescence, and the gravitational waves produced in these events are expected to be observed by gravitational wave detectors currently under construction. Gravitational wave interferometers such as LIGO (Abramovici et al. 1992) and VIRGO (Bradaschia et al. 1990) will soon be operational and will present the first opportunities to study NSMs via gravitational waves. Theoretical templates of the expected signal are required to extract signal information from the noisy background (Cutler et al. 1993). Post-Newtonian methods (Lincoln & Will 1990) may be adequate for the prediction of waveforms for the early stage of the inspiral. However, the prediction of waveforms in the later stages of the merger, when tidal effects and neutron star structure become important, requires a full three-dimensional numerical solution of the equations describing the motion of matter and the gravitational field.

In addition to gravitational wave astronomy, NSMs are of interest for nuclear astrophysical reasons. NSMs may yield information about the structure of neutron stars. Since the equation of state (EOS) of neutron star matter is not well constrained, the observation of gravitational wave signals from NSMs may provide constraints that could provide information about the dynamics of the merger and in turn the EOS of dense matter. Additionally, the material ejected during the coalescence of binary neutron stars may be a site of r-process nucleosynthesis (Lattimer et al. 1977). The r-process, which is thought to be responsible for the production of about 50% of elements heavier than iron in the universe, occurs when the capture rate of neutrons bombarding nuclei exceeds the beta decay rate. In the material ejected during NSMs there are expected to be regions where the r-process occurs robustly (Meyer 1989). Simulations of NSMs can allow us to study both the mass ejection and nucleosynthesis that occurs in the ejected material.

NSMs are a suggested source for the mysterious gamma-ray bursts observed by CGRO and other high-energy observational missions. NSMs are thought to release energy on the order of their gravitational binding energy $\approx 10^{53}$ erg, which may be larger than estimated gamma-ray burst energies $\approx 10^{51}$ (Quashnock 1996,Rees 1997) to 10^{53} erg (Woods & Loeb 1994). A popular model for bursts at cosmological distances is the relativistic fire-ball (Paczyński 1986, Goodman 1986, Shemi & Piran 1990, Paczyński 1990). NSMs are likely candidates for the source of relativistic fire-balls, but the mechanism by which the fire-ball develops has yet to be determined. Observations in the spring of 1997 of optical and X-ray counterparts to GRB 970228 (Costa, et al. 1997, Guarnierni 1997, Piro et al. 1997) and GRB 970508 (Bond 1997,

Djorgovski et al. 1997, Metzger et al. 1997), particularly the measurement of a Mg II absorption line at redshift $z = 0.835$ (Metzger et al. 1997), suggest that the bursts do indeed have a cosmological origin. Simulations of NSMs can test the consistency of the energetics and time scales with the estimated energies and observed time scales of the observed bursts.

One of the major difficulties in carrying out numerical simulations of binary neutron star mergers is developing a numerical algorithm that does not introduce unphysical dynamical effects into the problem. In order to avoid spurious inspiral, both the total energy and angular momentum must be conserved to sufficient accuracy. In the absence of physical instabilities or dissipative effects, such as gravitational radiation losses, the numerical methods should be capable of maintaining a binary neutron star system in a stable orbit. Additionally, when physical instabilities capable of causing coalescence are present, the algorithms must continue to conserve all important physical quantities. Without this capability it is impossible to develop quantitatively accurate models in situations where radiation losses are present. In this paper we consider several variations on the popular ZEUS hydrodynamic algorithm (Stone & Norman 1992) as applied to NSMs. The comparisons that will appear later in this paper examine the numerical effects of the choice of rotating versus inertial frames as well as the choice of several possible schemes for the coupling of gravity to the hydrodynamics. This paper is intended to lay the numerical groundwork for post-Newtonian and relativistic studies that will follow in later papers. While realistic models of NSMs are clearly relativistic or, at a bare minimum, post-Newtonian (PN) in nature, the examination of Newtonian models of orbiting binaries neutron stars is still of considerable value. Many of the lessons learned from Newtonian models will provide guidance for PN or GR modeling efforts. Indeed, if a numerical self-gravitating hydrodynamics algorithm is incapable of maintaining stable orbits for binary star systems in the Newtonian limit, then it is unlikely to be useful for more complex, and realistic, simulations of NSMs. In this paper we concentrate on purely Newtonian models of orbiting binary neutron stars in both the stable and unstable regimes. We will consider the evolution of initial configurations that are tidally unstable as well as initial configurations involving both spherical and "relaxed" neutron stars. Post-Newtonian models that make use of our numerical techniques will be considered in a subsequent paper.

1.2. Status of contemporary work on Newtonian and PN simulations of binary neutron star systems

The Newtonian and PN simulations that have been carried out to date can be placed into two categories: those that have employed Eulerian hydrodynamic methods (see Bowers & Wilson 1991 for a discussion of Eulerian methods) and those that have employed smoothed particle hydrodynamics (SPH) methods (see Gingold & Monaghan 1977 or Hernquist & Katz 1989 for a discussion of SPH methods), an inherently Lagrangean technique. Additionally, these simulations have been carried out in both inertial, i.e. laboratory, frames and in non-inertial, rotating frames. These simulations have also utilized a range of techniques for calculating the gravitational potential. Finally, these simulations have employed both spherical stars and equilibrium binary configurations as initial data. These choices can play a critical role in determining the outcome of the simulations. For this reason, in this section we briefly describe existing work on Newtonian and PN binary neutron star systems with a focus on the numerical techniques and the initial configurations that have been used. We first consider the Eulerian calculations followed by the SPH models.

The earliest Eulerian models of binary neutron star systems were carried out by Oohara and Nakamura (1989). This work and subsequent papers (Nakamura & Oohara 1989, Oohara & Nakamura 1990, Nakamura

& Oohara 1990, Nakamura & Oohara 1991, Oohara & Nakamura 1992) made use of purely Newtonian hydrodynamics, while later work with Shibata included PN effects (Oohara & Nakamura 1992, Shibata, Nakamura, & Oohara 1992,1993). As with all Eulerian, i.e. grid-based, hydrodynamics methods the underlying PDEs are discretized onto a coordinate mesh. The evolution of the mass distribution occurs as the material flows through the grid zones, and the equations governing the evolution are finite-differenced analogs of the Euler equations. There are many approaches to finite-differencing the Euler equations, but most modern formulations are at least second order in time and space, and have methods of realistically modeling shocks. The hydrodynamics method employed by the Oohara/Nakamura/Shibata calculations utilizes LeBlanc's method for transport, making use of a tensor artificial viscosity. A brief description is given in an appendix to Oohara and Nakamura (1989). The earlier calculations were carried out in the laboratory (fixed) frame, while later models utilized a rotating frame. In all of the calculations the gravitational potential was found by a direct solution of the Poisson equation. However, none of the papers discuss the boundary conditions for this equation. Finally, the papers have considered both spherical stars and equilibrium configurations for initial data although no comparisons of the two types of initial data were offered.

Ruffert et al. (1996,1997,1997) performed PN simulations of NSMs with the PROMETHEUS code implementing the Piecewise Parabolic Method (PPM) of Colella and Woodward (1984). The PPM is an extension of Godunov's method that solves the Riemann problem locally for the flow between zone interfaces, and, accordingly, it is well suited to addressing shocks. The calculation of the gravitational potential was accomplished by means of the direct solution of Poisson's equation using zero-padding boundary conditions (Hockney 1988). The initial conditions of the simulations were spherical neutron stars in both tidally locked and rotating configurations. The stars were embedded in an atmosphere of $10^9 \mathrm{g/cm}^3$ that covered the entire grid, and an artificial smoothing was performed on the surfaces of the stars to soften the edges. The earlier models considered configurations with the realistic equation of state of Lattimer and Swesty (1991) while later studies considered models with a much simpler polytropic EOS (Ruffert, Rampp, & Janka 1997).

The most recent Eulerian simulations of binary neutron star systems were performed by New and Tohline (1997). Their work focused on the evolution of equilibrium sequences of co-rotating, equal mass pairs of polytropes. The equilibrium sequences were constructed with Hachisu's Self-Consistent Field (SCF) technique (Hachisu 1986a,b). The dynamical stability of these equilibrium sequences was tested by evolving them with a 2nd order accurate finite-differenced Newtonian hydrodynamics code. The gravitational potential was obtained by a direct solution of Poisson's equation accomplished by means of the alternating direction implicit method. No description was given of the boundary conditions that were applied to the Poisson equation. The calculation was carried out in a rotating frame that was initially co-rotating with the binary system in order to avoid problems with the advection of the stars across the grid. In a stability test, a comparison of two white dwarf binary system simulations starting from the same initial conditions, one carried out in the inertial reference frame and the other carried out in the initially co-rotating frame, revealed dramatic differences in the dynamics of the binary system. This difference illustrates the need for very careful studies of purely numerical effects on these types of simulations. In the stability tests, New and Tohline found no points of instability for polytropic models with fairly soft equations of state ($\gamma = 2, 5/3$). They did find, however, an instability for the stiffer $\gamma = 3$ case indicating that systems with stiffer equations of state are susceptible to tidal instabilities. It is worthwhile to note that the authors state that they may have misidentified some stable systems as unstable had they performed their simulations in the inertial reference frame, and they therefore stress the importance of very careful studies of numerical effects and careful comparison of different numerical methods.

The earliest SPH simulations of binary neutron star systems began with the work of Rasio and Shapiro (1992,1994,1995). In SPH, a distribution of mass in a particular region of space is represented by discreet particles such that the mass density of the particles is proportional to the specified density of the fluid. Local calculations of fluid quantities are accomplished by smoothing over the local distribution of particles. The method is inherently Lagrangean. This numerical work accompanied semi-analytical work with Lai, with the goal of predicting the onset of instabilities in binary systems (Lai, Rasio, & Shapiro 1993a,1993b,1993c,1994a,1994b, 1994c). The calculation of the gravitational potential was accomplished by mapping the particle distribution onto a density distribution on a mesh where the Poisson equation was solved by means of fast Fourier transform methods. In these calculations, the authors employed several different polytropic equations of state. The equilibrium initial data for the models were obtained by allowing the spherical stars to relax to a steady state solution in a rotating frame of reference. The work investigated sequences of binary star systems with a range of initial separations, and their construction of equilibrium initial configurations for evolutions was critical in determining if and when a dynamical instability forced the merger. The group reported the presence of a dynamical instability at separations that increase with an increase of the polytropic exponent γ.

The Drexel group performed Newtonian SPH simulations of nonaxisymmetric collisions of equal mass neutron stars (Centrella & McMillan 1993). This effort was followed by calculations of Newtonian NSMs by Zhuge et al. (1994,1996) They employed the TREESPH implementation of SPH developed by Hernquist and Katz (1989) in which the gravitational forces are calculated via the hierarchical tree method of Barnes and Hut (1986). The calculations employed a polytropic EOS with $\gamma = 5/3$ and $\gamma = 2$. These calculations were carried out in the laboratory frame and the initial data consisted of spherical stars in the non-rotating case or rotating stars that were produced using a self-consistent field method. The work was aimed at studying the gravitational radiation emission from NSMs and addressed effects due to the equation of state, spins, and mass ratio of the stars on the gravitational wave energy spectrum. The coalescence in these calculations was driven by a frictional force term added to the hydrodynamic equations that models the effects of gravitational radiation loss.

Davies, Benz, Piran, and Thielemann (1994) performed SPH simulations of NSMs with a focus on the nuclear astrophysical and thermodynamic effects of coalescence. The SPH code used for these calculations was described in earlier work on stellar collisions (Benz & Hills 1987), and it makes use of a tree algorithm for calculating gravitational forces. The calculations were carried out in the inertial frame. The initial data for the neutron stars was modeled as equal mass $\gamma = 2.4$ polytropes, but a more realistic EOS was employed for the dynamical calculation. The driving force behind the coalescence was a frictional force model of gravitational radiation loss similar to that of the Drexel group. The rates of energy and angular momentum loss were determined by applying the quadrupole approximation to the equivalent point mass system, and the resulting acceleration of each SPH particle was determined by expressions derived from these rates. Subsequent calculations by Rosswog et al. (1998) have focused on r-process nucleosynthesis and mass ejection. More recently members of this group have developed a PN extension of the SPH algorithm (Ayal et al. 1999), which they have applied to NSMs to study the dynamics and gravitational wave emission from the merger.

1.3. Outline

In the remainder of this paper we will focus on comparisons of several Eulerian numerical methods for modeling Newtonian binary neutron star systems as well as comparisons of the effects of spherical versus

equilibrium initial data. Additionally, we will examine the stability of both the spherical and equilibrium initial data. The subject of gravitational wave signals from NSMs will not be considered here, but instead will be the subject of a subsequent paper.

In §2 we explain our numerical methods for evolving the equations of hydrodynamics and solving the Poisson equation. In §3 we delineate our method for obtaining equilibrium data with self-consistent boundary conditions. In §4 we compare calculations carried out in both rotating and inertial frames with several different schemes for coupling gravity to matter via the gas momentum equations. In §5 we compare models with both equilibrium and non-equilibrium initial data. In §6 we offer conclusions about this work especially regarding its meaning for post-Newtonian and fully general relativistic models of neutron star mergers.

2. Numerical Hydrodynamics Algorithms

As we have mentioned in the previous section, the accuracy of the numerical algorithms is of paramount importance if one is to obtain accurate hydrodynamic models for mergers. Previous work on such models have utilized a variety of hydrodynamic schemes in either a fixed (inertial) or rotating frame of reference. The latter has been claimed to be more accurate by virtue of is obviation of the difficulties of advection, but no systematic comparison of the two has yet been published. In this section we describe two numerical hydrodynamics schemes that we have employed to produce such a study. We have not attempted an exhaustive study in which we compare the qualitative results of each hydrodynamic scheme that has been employed to date. Such studies have been conducted for most of these schemes on a number of problems involving shocks. However, the performance of the hydrodynamic algorithm on shocks is not the only metric by which one needs to measure the quality of the hydrodynamic algorithm. For example, for simulations in which the linear momentum equation has been solved one should examine how well angular momentum is conserved. In the long timescale evolutions needed for multiple orbit simulations of orbiting stars, the addition or loss of angular momentum into the calculation could artificially enhance or delay inspiral during the mergers. Similar issues apply for linear momentum in simulations where the gas angular momentum equations are solved. In the same vein, if the gas momentum equation is solved then one should monitor how well the total energy is conserved. Or if the total energy equations is solved one should monitor how well the gas energy equation is solved.

The 3-D numerical hydrodynamics scheme we describe in the section is similar to the ZEUS scheme of Stone & Norman (1982). However, we have made some fundamental changes to the order of operations in order to improve the numerical accuracy of the scheme on self–gravitating problems.

2.1. Euler Equations

The flow of matter in the neutron stars can be taken to be inviscid. Under these circumstances the Newtonian description of the matter evolution is described by the continuity equation together with the Euler equations (Mihalas & Mihalas 1984, Bowers & Wilson 1991) of compressible inviscid hydrodynamics. In an inertial frame of reference the equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (E\mathbf{v}) = -P\nabla \cdot \mathbf{v} \tag{2}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla \cdot (\rho v_i \mathbf{v}) = -\left(\nabla P\right)_i - \rho\left(\nabla\Phi\right)_i, \tag{3}$$

where the dependent variables are the mass density ρ, the internal energy density E, the fluid velocity v_i, the fluid pressure P, and the Newtonian gravitational potential Φ. The gravitational potential is described by the the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho \tag{4}$$

in conjunction with boundary conditions that must be specified.

In a frame rotating with angular frequency $\boldsymbol{\omega}$ about the center of the fixed (inertial) grid the gas momentum equation is modified by the addition of the Coriolis and centrifugal forces (Chandrasekhar 1969)

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla \cdot (\rho v_i \mathbf{v}) = -\left(\nabla P\right)_i - \rho\left(\nabla\Phi\right)_i - 2\rho\left(\boldsymbol{\omega} \times \mathbf{v}_r\right)_i - \rho\left(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right)_i \tag{5}$$

where $\mathbf{v}_r$ is the velocity of the rotating frame relative to the lab frame. The notation $(\mathbf{A})_i$ indicates the ith component of the vector $\mathbf{A}$. In the limit of $\omega \longrightarrow 0$ we recover the inertial frame momentum equation. The continuity, gas energy, and Poisson equations are unchanged from the inertial frame case.

For the remainder of this paper we will consider the angular frequency vector $\boldsymbol{\omega}$ to be co-aligned with the z axis so that

$$\boldsymbol{\omega} = \omega\hat{\mathbf{e}}_z. \tag{6}$$

We assume that the z axis passes through the center of the grid at coordinates (x_c, y_c). Under this condition the Coriolis and centrifugal force terms in Cartesian coordinates become:

$$\left(\frac{\partial\rho\mathbf{v}}{\partial t}\right)_{\rm cc} = 2\omega\rho\left(v_y\hat{\mathbf{e}}_x - v_x\hat{\mathbf{e}}_y\right) + \omega^2\rho\left((x - x_c)\hat{\mathbf{e}}_x + (y - y_c)\hat{\mathbf{e}}_y\right), \tag{7}$$

where x_c and y_c are the coordinates of the z-axis.

The set of hydrodynamic equations must be closed by specifying an equation of state expressing pressure as function of local thermodynamic quantities. A standard choice for building the initial neutron star models is the polytropic equation of state, which has the form

$$P = (\gamma - 1)\, E \tag{8}$$

where γ is the polytropic exponent, which is related to the polytropic index, n, by the relationship

$$\gamma = 1 + \frac{1}{n}. \tag{9}$$

This particular type of EOS is advantageous in that the gas energy equation becomes linear in E rendering the solution trivial. In isentropic situations this EOS allows the pressure to be written purely as a function of density in the form

$$P = K\rho^\gamma, \tag{10}$$

where P is the pressure, and K is the polytropic constant. This form of the EOS is used for the construction of the initial models.

2.2. Numerical Solution

Much of the numerical scheme we employ for the solution of the Euler equations is derived from the ZEUS-2D hydrodynamics scheme invented by Stone & Norman (1992) (hereafter SN). In particular the finite-differencing stencils are identical to those of SN, with the exception of the Coriolis and centrifugal forces, which are not included in the ZEUS-2D scheme. However, the method we employ differs in one significant way: the order of solution of the various terms in the Euler equations differs from that of the ZEUS-2D algorithm. As we will show in a subsequent section of this paper, the order of the solution of these equations is of fundamental importance to the accuracy of the algorithm in the case of self-gravitating hydrodynamics. A final simplification for our algorithm, which we shall henceforth refer to as the V3D algorithm, employs only Cartesian coordinates. This simplification significantly increases the computational speed of the V3D code.

The finite-differencing algorithm we employ relies on a staggered grid in which the intensive variables E, and ρ are defined at cell centers, while the vector variables such as the velocity components v_i are considered to be defined at their respective cell edges. The gravitational potential Φ is also defined at the cell center. The centering of the variables on the grid is depicted in a 2-D plane of the 3-D grid in Figure 1. In our finite-difference notation we employ superscripts to denote the time at which the variables are

Fig. 1.— A diagram of a 2-D slice of the staggered mesh illustrating the locations at which the variables are defined. The plane shown is located at a "z" coordinate $z_{k+\frac{1}{2}}$.

defined. The timestep is taken to be Δt and the numerical algorithm advances the solution of the PDEs from nth time, t^n, to the new $(n+1)$th time, $t^{n+1} = t^n + \Delta t$.

The explicit finite-difference algorithm employed by SN for the solution of the Euler equations decomposes the time integration into two steps by employing operator splitting among the various terms of the equations. In one step the density, internal energy density, and velocities are updated by integrating the advective terms. In the nomenclature of SN we refer to this as the *transport* step. The remaining terms, i.e. the terms on the right hand sides of equations (1), (2), and (5), are integrated forward in time. Following SN we refer to this step as the *source* step.

An additional consideration involves the solution of the Poisson equation and how it relates to the solution of the Euler equations. In the ZEUS-2D algorithm the order of solution of the Euler equations is described in the flow chart of Figure 2a. In contrast our algorithm, V3D, is described in Figure 2b.

Fig. 2.— The order of operations to update the hydrodynamic and gravitational potential variables from time t^n to time t^{n+1}.

In the transport step the following equations are solved:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \tag{11}$$

$$\frac{\partial E}{\partial t} = -\nabla \cdot (E\mathbf{v}) \tag{12}$$

$$\frac{\partial (\rho v_i)}{\partial t} = -\nabla \cdot (\rho v_i \mathbf{v}). \tag{13}$$

The advection integration is carried out using Norman's consistent advection scheme (Norman 1980), which ties the advected internal energy density and advected velocity to the mass flux as described in SN. This concept of tying the energy and momentum fluxes to the mass flux has been shown to posses superior angular momentum conservation properties (Norman 1980). The actual flux limiter employed is the van Leer monotonic flux limiter (van Leer 1977) that is spatially second order.

In the source step the following equations are updated:

$$\frac{\partial E}{\partial t} = -\left(P + Q\right) \nabla \cdot \mathbf{v} \tag{14}$$

$$\frac{\partial (\rho v_i)}{\partial t} = -\left(\nabla P\right)_i - \left(\nabla Q\right)_i - \rho \left(\nabla \Phi\right)_i - 2\rho \left(\boldsymbol{\omega} \times \mathbf{v}_r\right)_i - \rho \left(\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right)\right)_i . \tag{15}$$

The scalar viscous stress Q is added to the equations in order to allow for viscous dissipation by shocks in the fluid. We employ the standard von Neumann-Richtmyer prescription for the viscous stress, as described in SN, with a length parameter of $\ell = 2$. We monitor the viscous dissipation arising from this stress and have found that the total viscous energy generation is negligible for two merging polytropes. The coupling to gravity enters through the gradient of the Newtonian gravitational potential in equation (15).

We wish to note that the continuity equation is not updated during the source step as it possesses no source term. The lack of a source term for the continuity equation means that the density at a new time t^{n+1} is known after the transport step is complete. As we will discuss in a subsequent section of this paper, this point is crucial to our preferred method of solution for these equations.

The explicit finite-differencing of the Euler equations is briefly discussed in appendix A. For the remainder of this section we concentrate on the order of solution of the transport and source steps. In the method of SN the source terms (equations 14 and 15) are integrated forward in time to arrive an intermediate solution for the new internal energy density E and the velocity $\mathbf{v}$. The intermediate energy density and velocity are used as initial values for the transport equations (11)-(13), which are then integrated forward in time to find the values of the density, energy density, and velocity at t^{n+1}. SN have shown by means of convergence testing that this algorithm is spatially second order accurate.

There is no compelling reason to suggest that the order of source and transport updates as presented by SN is preferred. One can easily reverse the order of updates so that the results of the transport step are utilized in the source update. We henceforth will refer to this order of updates as the V3D algorithm while the opposite order will be referred to as the ZEUS algorithm. By comparing the two algorithms on a number of standard hydrodynamic test problems we have numerically verified that the reversal of these operations has no significant effect on the overall quality of solutions when self-gravity is not present. However, as we will show in a later section the V3D algorithm offers significant advantages when modeling orbiting binary stars.

An example of the comparable performance of the ZEUS and V3D algorithms on a standard test problem is shown in Figure 3 where the performance of the algorithm on a Sod-like (Sod 1978) shock tube is shown. The shock tube problem pictured employs a $\gamma = 5/3$ polytropic equation of state. The grid is set up with 100 spatial zones over the range of $-2 < x < 2$ cm with the initial contact interface at $t = 0$ located at $x = 0$. This initial configuration is that of a Riemann problem, which results in a shock

Fig. 3.— The numerical solution of a shock tube problem for the ZEUS (cross symbols) and V3D (circles) algorithms. The solid line illustrates the exact Riemann solution to this problem.

and a contact discontinuity propagating to the right and a rarefaction propagating to the left. Because the exact solution is known (Chorin & Marsden 1993) we can easily evaluate the numerical results from both algorithms. Overall the character of the numerical solution is comparable between the two cases. The values of the variables in both the contact discontinuity and the shock are slightly different as would be expected from different algorithms, but both methods resolve the shock and the contact discontinuity with the same number of zones. The rarefaction is represented nearly identically by both methods. The figure compares the two orders of update. One can visually see that little difference exists between the two solutions. We have verified this on a number of other non-self-gravitating test problems. In contrast, for the case of self-gravitating hydrodynamics, we do find that the V3D algorithm is preferred as we will discuss in section 4.

2.3. Numerical Solution of the Poisson Equation

In order to describe self-gravitating phenomena, the gravitational field of the matter distribution must be found by solving the Poisson problem described by equation (4). The Poisson equation can be readily solved by a variety of techniques well suited to elliptic equations. In the simulations described in this paper we have employed both W-cycle multigrid and Fast-Fourier-Transform (FFT) methods (Press et al. 1992). Both methods have been extensively tested on matter configurations where the solution is known. Because the Poisson equation is linear one can easily generate test problems with known answers, but which also posses complex field geometries. For example, by placing $\gamma = 2$ polytropes at random points within the computational domain we can create a complex gravitational field configuration. Since the gravitational potential for a $\gamma = 2$ polytrope is analytically known, the potential for the entire configuration at any point in the computational domain is readily found as a superposition of individual polytropic solutions. Using this method we have found that both methods give the correct answers to approximately $\sim 10^{-5}$ for the grid resolutions employed in this work.

The numerical solution of equation (4) requires the specification of boundary values along the edge of the computational domain. For the problem of merging neutron stars these are *a priori* unknown. The problem of determining the appropriate boundary conditions has been approached differently by a number of different groups. Ruffert et al. have employed zero-padding boundary conditions in conjunction with their FFT solution method. The zero padding method has been shown by James (1977) to be algebraically equivalent to a direct summation by convolution of image charges (defined on the edge of the grid) over the Green function for the Poisson equation. Oohara and Nakamura (1990), Rasio and Shapiro (1992,1994,1995), and New and Tohline (1997) have not specified how boundary conditions on the potential were obtained for their hydrodynamic solutions. A number of groups (Davies et al. 1994; Zhuge et al. 1994; Zhuge et al. 1996) have carried out smoothed particle hydrodynamics (SPH) simulations in which the field was computed by a tree-code summation obviating the need for the specification of boundary values.

The accurate specification of self-consistent boundary conditions for the Poisson equation is challenging. The expansion of the potential in terms of multi-poles may require the expansion to be carried out to very

high order if one is to obtain accurate values for the potential. This is especially true given that the initial configuration for a neutron star merger simulation consists of two widely separated fluid bodies, which has a very large quadrupole moment. The zero-padding method necessitates a fairly large memory cost. This memory cost can largely be eliminated by use of the James algorithm for the solution of the Poisson problem for isolated systems (James 1977). In fact we employ the James algorithm to obtain equilibrium initial data described in the next section. However, we have found that the James algorithm does not scale well to large numbers of processors on shared-memory parallel computers.

In order to obtain an accurate algorithm for the boundary conditions on the potential we have turned to direct integration over the Green function for Poisson's equation, i.e.

$$\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'. \tag{16}$$

Because of the large number of grid zones present in the problem, it is computationally intractable to compute this sum directly. If our computational domain is discretized into N points in each of the three spatial dimensions then the summation is over N^3 zones in order to evaluate the potential at each of the $6N^2$ points on the edge of the domain. This implies the total algorithm is order N^5. Since we typically employ $N = 128$ for our simulations, this renders the direct summation over the entire grid computationally intractable for use in a hydrodynamic simulation. This dilemma is further exacerbated by the need to calculate an inverse square root in order to evaluate the distance r from the boundary point to each zone. However, for the problems we are considering the mass is concentrated in only a relatively small region of the computational domain. If we restrict the summation to only those zones in which a significant amount of mass is present, the summation becomes more tractable. Accordingly, we have adopted the following algorithm for obtaining the boundary values.

We evaluate the amount of mass in eight-zone cubic ($2 \times 2 \times 2$) "blocks" of the grid. If the mass of the zone is greater than a threshold value $M_{\rm th}$ the total mass of the block and the blocks center of mass coordinates are stored in a list. This operation is order N^3 but it is only required once per timestep. Once the entire mesh has been scanned we obtain a complete list of all the block with a significant mass. The size of the list is dependent on the mass threshold employed. If the mass threshold is chosen too low, the list will become very large and the summation computationally intractable. If the cutoff is chosen too large, the list will not include most of the mass in the domain. We have experimentally found that a value of $M_{\rm th} = 10^{-5} M_\odot$ produces a list which fully represents the mass in the domain. Once the list of significant mass blocks has been produced the boundary values of the potential can be calculated by direct summation

$$\phi_{ij} = \sum_{k=1}^{k=L} \frac{M_k}{r_{ijk}}, \tag{17}$$

where M_k is the mass in the kth block in the list, r_{ijk} is the distance from the ijth point on the edge of the grid to the center of mass coordinates of the kth block, and L is the length of the list. This operation is order $N^2 L$. However, $L \ll N^3$, which renders the summation tractable. Furthermore, this algorithm is readily parallelizable on a shared memory parallel computer thus allowing for a rapid solution. We typically find that calculation of the boundary conditions never exceeds 20% of the overall computational effort. Finally, we wish to emphasize that this algorithm will not work efficiently in cases where the mass is more evenly distributed over the entire mesh.

The accuracy of this algorithm has been tested by two methods. First, the algorithm was applied to the test problems we mentioned earlier in this section where the analytic answer was known. Secondly,

the summation algorithm was also tested in more general situations by comparing the boundary values obtained by this method to those obtained by brute force direct summation. In all cases the boundary values agreed to better than 10^{-4}; in most cases the agreement was better than 10^{-5}. Additionally, we track the total mass in the list so that it can be compared to the total mass on the mesh. Any significant difference between the two masses will indicate a problem with the summation.

3. Equilibrium Initial Data

3.1. Numerical Methods for Obtaining Equilibrium Data

In order to accurately model two neutron stars in close orbits with one another, it is important to employ initial conditions that precisely reflect the true configurations of the two fluid bodies. In general, for close binary systems, these equilibrium configurations will not consist of two spherical stars. Instead the configuration will contain tidally distorted fluid bodies that are only approximately spherical. In numerical simulations of binary star systems the stability of the orbits can be quite sensitive to the details of the initial configuration. In a subsequent section we compare dynamical models which have employed equilibrium initial conditions with models that have utilized spherical stars.

The construction of initial data for these systems is a non-trivial task. In practice each neutron star in a binary system will be non-synchronously rotating around it's own axis. Several calculations (Bildsten & Cutler 1992; Kochanek 1992) have shown that viscous dissipation at the causal limit is insufficient to tidally lock binary neutron star systems during their lifetime. Accordingly, the most realistic configurations that one could model would be non-tidally locked. However, there is tremendous difficulty of obtaining equilibrium initial data for such cases. Finding initial conditions that correspond to the non-synchronous case would require the solution of the compressible Darwin–Riemann problem that is well outside the scope of this paper. Since the target of this paper is a study of the numerical methods and initial conditions needed for precise simulations of binary neutron star mergers, we restrict ourselves to the tidally locked case. Realistic binary systems will also contain unequal mass components. However, in this paper we consider only equal mass systems. Our numerical algorithm for obtaining equilibrium initial data is easily extended to the non-equal mass case which will be considered in a future paper.

A number of other research groups (Oohara & Nakamura 1990; New & Tohline 1997) have developed methods to obtain equilibrium data for the case of synchronous binary neutron star systems. In both cases the equilibrium models must simultaneously satisfy both the Bernoulli and Poisson equations. In the case of Oohara and Nakamura they have employed a method that in similar to ours in that it iteratively solves the Bernoulli and Poisson equations on a Cartesian grid. However, we have found some problems with this method for obtaining the initial conditions that we seek. New & Tohline have found initial conditions using the self-consistent field technique of Hachisu (1986a,1986b), which iteratively solves the Bernoulli equation and the integral form of the Poisson equation on a spherical polar grid. While this latter method avoids the problems we have found with the Oohara method, it's use for our case would involve remapping of the data from the polar grid to the Cartesian grid we employ for our dynamical simulations. This remapping would introduce small errors that would render the initial conditions on the Cartesian grid slightly out of equilibrium. In turn, the deviation from equilibrium can cause spurious hydrodynamic motions away from the initial data once the evolution begins. In order to avoid this we have combined techniques from both the Oohara et al. and New & Tohline methods. Our objective is to develop equilibrium data on the same grid that the hydrodynamic simulation will employ.

We first need to identify the equations that describe the equilibrium configuration. These equations result from taking the hydrostatic limit of the Euler equations of hydrodynamics together with the Poisson equation. In the hydrostatic limit the gas momentum equation (5) collapses to

$$\nabla \cdot (\rho v_i \mathbf{v}) + (\nabla P)_i + \rho (\nabla \Phi)_i + 2\rho (\boldsymbol{\omega} \times \mathbf{v}_r)_i + \rho (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}))_i = 0. \tag{18}$$

If we make the assumption that the equation of state is of the isentropic form given by equation (10), then equation (18) can be integrated by parts to find the Bernoulli equation

$$\frac{\gamma K \rho^{\gamma-1}}{\gamma - 1} + \Phi + \frac{\omega^2}{2} \left[(x - x_c)^2 + (y - y_c)^2 \right] = C \tag{19}$$

where C is a constant and where we have assumed that the rotation is about the z-axis. This equation must be satisfied in the interior of the fluid bodies. Simultaneously the equilibrium data must also satisfy the Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho. \tag{20}$$

However, two fundamental difficulties occur in the solution of these two equations on Cartesian grids containing self-gravitating fluid bodies. First, we do not *a priori* know where the boundaries of the fluid bodies lie, and hence we do not know where the Bernoulli condition should apply. Second, we do not *a priori* know the boundary conditions on Φ that must apply to the Poisson equation. The boundary conditions on Φ must be determined in a self–consistent fashion using the Green function corresponding to the Poisson equation. This latter problem is the more difficult of the two problems to solve. While Oohara et al. have employed a direct solution of the Poisson equation for equilibrium data, they have made no mention of what boundary conditions they have employed on equation (20). We have found that the configuration resulting from the iterative solution of equations (19) and (20) is quite sensitive to the use of non-self-consistent boundary conditions and we strongly recommend against employing such boundary conditions.

In order to minimize the problems with deciding where the boundaries of the fluid bodies are, we have adopted a technique from the SCF technique of Hachisu et al. (1990). We consider equilibrium binary systems in two different topological configurations as depicted by Figure 4. In the first case we consider non-contact binary systems. In the second case we consider contact binary systems. In the first case during our iterative solution of the combined Bernoulli and Poisson equations we specify the extremal inner and outer points of the star as depicted in Figure 4A. We define the orientation of our grid so that the x-axis passes through the centers of mass of the two stars. The z-axis passes through the barycenter of the system thus defining the origin of the grid. By specifying the inner and outer points we seek equilibrium solutions with a certain aspect ratio. By adjusting the locations of the extremal points we can find equilibrium configurations with varying separations between the components and or their centers of mass. In the case of contact binaries we specify the extremal outer point of the contact system and the extremal outer point of the neck connecting the high density portions of the two fluid bodies. As with the detached case, by varying these two points we can find a sequence of equilibrium configurations with varying separations between their centers of mass. In the contact binary case we define the center of mass of each star by considering only the mass contained within each half of the computational domain as defined by a plane perpendicular to the line connecting the two highest density zones of the grid.

Once we have identified the configuration of the system we can then determine where the Bernoulli equation can be applied during each iterative step. Assuming, for the moment, that we know ω, K, and C, and that we posses some iterative estimate of Φ, when can then solve the Bernoulli equation for a new

Fig. 4.— Separated and contact binary topologies

estimate of the density ρ:

$$\rho = \left[\frac{\gamma - 1}{\gamma K} \left(C - \Phi - \frac{\omega^2}{2} \left[(x - x_c)^2 + (y - y_c)^2 \right] \right) \right]^{1/(\gamma - 1)} . \tag{21}$$

Obviously, this equation only makes sense where the factor contained within the square brackets of equation (21) is positive. We use this as a criterion to decide where to apply the Bernoulli equation. If the factor is positive the density is updated to the density as determined by equation (21), otherwise the grid zone is considered to be vacuum and the density is set equal to zero.

The determination of self-consistent boundary conditions for the Poisson equation is of paramount importance. In order to clarify what we mean by the use of the term "self-consistent" we first clarify the problem. We assume that the computational domain, Ω, contains the self-gravitating fluid bodies which have compact support within the interior of the domain, i.e. the fluid density vanishes on the boundary of the domain $\partial\Omega$. More simply stated, we assume that the fluid bodies are contained inside the domain Ω. This type of self-gravitating system has been termed an "isolated" system (James 1977). Under these circumstances we require that the boundary conditions on Φ satisfy equation (20). Since the potential on the boundaries depends on the density distribution $\rho(\mathbf{r})$ we cannot *a priori* self-consistently know the boundary values prior to solving the problem. This problem is not only relevant for initial data but is also relevant for the solution of the Poisson problem during the course of a self-gravitating hydrodynamic simulation. Various groups modeling equilibrium binary configurations have attempted to avoid this problem. The equilibrium sequence work of NT has utilized the SCF method of Hachisu, which avoids this problem by employing a multi-pole expansion of the potential in order to estimate the boundary conditions.

In the case of isolated systems, James (1977) has shown that the boundary conditions can be obtained exactly by use of FFT techniques. We accordingly employ this method for use in our initial data algorithm. Furthermore, James has shown that this method is algebraicly equivalent to the "zero-padding" technique employed by Ruffert et al. (1996). The advantage of the James algorithm over the zero-padding technique is that it requires substantially less memory overhead which is a significant advantage in a 3-D simulation. We could also employ this algorithm to compute the self-consistent potential during the course of our hydrodynamic simulations. However, in the hydrodynamic simulations we have found it advantageous to employ equation (16) directly to get the boundary conditions for Φ followed by a straightforward Poisson solve using either multigrid or FFT techniques (Press et al. 1992). We have found that this method is more amenable to implementation on the shared memory parallel computing architectures that we employ for our simulations.

The complete algorithm for finding the initial data is as follows:

1. Fix inner and outer points of stars (in the detached binary) case and outer point and neck width (in contact binary case). Denote the distance from the z-axis to these points as $R_{\rm out}$ and $R_{\rm in}$.
2. Make initial guess at the density distribution throughout the computational domain. Also guess an initial value of K.

3. Using density distribution solve for potential using the James algorithm to solve the Poisson equation with self-consistent boundary conditions.

4. Using the Bernoulli equation evaluate ω by

$$\omega^2 = \frac{2(\Phi_{\rm out} - \Phi_{\rm in})}{R_{\rm out}^2 - R_{\rm in}^2} \tag{22}$$

5. Evaluate the Bernoulli constant, C, at $R_{\rm out}$ using equation (19)

6. Update K by evaluating the Bernoulli equation at some point x which lies on the line through the centers of the stars

$$K = \frac{\gamma - 1}{\gamma} \rho^{\gamma - 1}(x) \left(C - \Phi(x) + \omega^2 (x - x_c)^2 \right) \tag{23}$$

7. Calculate new value of density for every zone on the grid using the following algorithm:

$$\rho(\mathbf{x}) = \begin{cases} \chi(\mathbf{x})^{1/(\gamma-1)} & \text{if } \chi(\mathbf{x}) > 0 \\ 0 & \text{if } \chi(\mathbf{x}) \leq 0 \end{cases} \tag{24}$$

where

$$\chi(\mathbf{x}) \equiv \frac{\gamma - 1}{K\gamma} \left(C - \Phi(\mathbf{x}) + \omega^2 \left[(x - x_c)^2 + (y - y_c)^2 \right] \right). \tag{25}$$

8. If the maximum relative density change in any zone of the grid is less than 10^{-5} then consider the solution converged and stop. Otherwise go to step 3.

One major difference between this algorithm and those utilized by others is step 6, the update of K. For a particular equation of state, e.g. $\gamma = 2$, there may not be a solution for an equilibrium configuration with a given inner and outer point. One can easily see this for the case of an isolated $\gamma = 2$ polytrope where the radius is determined by (Shapiro & Teukolsky 1983)

$$R = \left[\frac{K}{2\pi G} \right]^{1/2}. \tag{26}$$

In this case only a specific value of K will allow the star to "fit" into the specified number of grid points between $R_{\rm out}$ and $R_{\rm in}$. If the value of K is not allowed to change the iterative procedure described does not converge. In practice the change in K is small.

When constructing an equilibrium sequence with fixed values of K, we adjust the grid size slightly to get the desired value of K. We have found that this usually only requires changes of a few percent in the grid spacing Δx in order to find an equilibrium solution for a specified value of K. The total mass of the converged equilibrium system is determined by the initial guess of the density distribution in step 2 of the algorithm. By multiplying the initial guess of the density distribution by some factor we can converge to equilibrium systems of more or less mass. For the case of $\gamma = 2$ both the Bernoulli and Poisson equations are linear in the variables Φ and ρ. In this case the total mass of the converged solution is affected only by the initial guess at the distribution while the value of K is determined only by the grid spacing. This renders the procedure of producing a sequence of equilibrium solutions for a given polytropic constant, K, and total mass, M_T, relatively easy. In the case where $\gamma \neq 2$ the Bernoulli equation becomes non-linear and changes in Δx or the initial density guess affect both the resulting value of K and M_T. In this case building an equilibrium sequence becomes much more difficult and time consuming. For this reason we have constructed a $\gamma = 3$ equilibrium sequence only at $65 \times 65 \times 65$ resolution. We have constructed a few specific $\gamma = 3$ equilibrium configurations at $129 \times 129 \times 129$ resolution for use in hydrodynamic studies of stability.

Fig. 5.— The total energy and angular momentum for equilibrium data obtained using the James method for $\gamma = 2$ ($n = 1$). The crosses indicated models constructed on a 65^3 grid while the squares indicate models constructed on a 129^3 grid.

Fig. 6.— The total energy and angular momentum for equilibrium data obtained using the James method for $\gamma = 3$ ($n = 0.5$). All models were constructed on a 65^3 grid.

3.2. Equilibrium Sequences of Initial Data

Using the method that we have described above we have constructed equilibrium sequences of data for $\gamma = 2$ ($n = 1$) and $\gamma = 3$ ($n = 0.5$) polytropes. For the $\gamma = 2$ case, the sequences were constructed with a total mass of $M_T = 2.8 M_\odot$ and a value of $K = 4.196 \times 10^4$ erg cm^3 g^{-2}. An isolated spherical $\gamma = 2$ polytrope with these parameters would have a radius of approximately $R \approx 10$ km and a central density that is roughly 10 times the nuclear saturation density of $\rho_s = 2.5 \times 10^{14}$ g/cm^3. Such a configuration resembles a realistic neutron star. The $\gamma = 2$ sequence is shown in Figure 5 while the $\gamma = 3$ case is shown in Figure 6. All separations shown are the center of mass separation $a_{\rm cm}$ which has been normalized to the spherical radius of a single undisturbed polytrope. Both the total energy, $E_{\rm tot}$, and the angular momentum, J, are plotted for each configuration.

In the $\gamma = 2$ sequence the models with a separation of less than about $a_{\rm cm} = 2.8$ are contact binaries where the two stars are joined by a "neck" of matter passing through the barycenter of the system. Those systems with separations greater than $a_{\rm cm} > 2.8$ are detached. For the $\gamma = 3$ case the bifurcation point is at a separation of approximately $a_{\rm cm} = 3$. In order to obtain this number more precisely we would have to construct models at substantially higher resolution. Because of the difficulty of constructing a large number of configurations with a specified value of K and M_T for the non-linear $\gamma = 3$ case, we have chosen not to do so. Our purpose was to construct initial data for hydrodynamic simulations using the same grid that we would employ for the simulation.

For the models shown in Figures 5 and 6 the grid resolution was approximately $\Delta x = 1.0$ km for the 65^3 models and $\Delta x = 0.5$ km for the 129^3 models. From Figure 5 it is easily observed, by comparing the 65^3 and 129^3 models, that the 65^3 models do not have adequate spatial resolution at the wider separations. Nevertheless both the 65^3 and 129^3 models show minima in both the total energy, $E_{\rm tot}$, and angular momenta, J, at approximately $a_{\rm cm} \approx 2.8$. The slight variation in the data near the bifurcation point between detached and contact binaries is due to the finite resolution of the grid. The contact binaries in this case may have a neck consisting of only one or two zones, a situation which is likely to cause some fluctuation in both the energy and the angular momentum due to the discrete nature of the neck.

It is interesting to compare these results with the semi-analytic work of Lai et al. (1993c) (hereafter LRS). LRS constructed an equilibrium sequence of binary, compressible, Darwin ellipsoids as an approximation to the equilibrium configurations of two synchronously orbiting polytropes. By identifying a turning point in the energy versus separation curves LRS found a secular instability for $\gamma = 2$ polytropes at a separation of $a_{\rm cm} = 2.76$. Furthermore, LRS also found that these turning-points occured simultaneously

in both the total energy, E, and the angular momentum, J.

There have also been a number of efforts to construct such equilibrium sequences numerically. In addition to their semi-analytic work LRS also found equilibrium sequences obtained using a relaxation scheme which employed smooth particle hydrodynamics methods yielded a turning point in the energy and angular momenta for an equilibrium sequence at a separation of $a_{\rm cm} = 2.9$. In contrast New and Tohline (1997), using the SCF technique, found a turning point at $a_{\rm cm} = 2.98$. Our result, which can be readily seen from Figure 5, yields an approximate turning point of $a_{\rm cm} \approx 2.85$. This result is somewhat closer to the compressible Darwin ellipsoid value and much further from the recently obtained value of New & Tohline. In agreement with both LRS and NT, we find the turning point at a point where the equilibrium systems are still detached. Nevertheless, the turning point is quite close to the point at which attached systems would form. The occurrence of a turning point on the detached binary branch of the curve seems to indicate that a binary system slowly spiraling inward by some energy and angular momentum loss mechanism will encounter an instability without ever becoming a contact binary. The nature of this instability and its implication for the dynamical evolution of binary systems will be discussed in a subsequent section.

In the $\gamma = 3$ case our results indicate a turning point in the equilibrium sequence. In this case the value of the polytropic constant was $K = 4.961 \times 10^{-11}$ erg cm^6 g^{-3} and the total mass was $M_T = 2.8 M_\odot$. Note that in this case the turning point we seem to find is at approximately $a_{\rm cm} \approx 3.05$ in comparison with the LRS semi-analytic value of $a_{\rm cm} = 2.99$. In contrast NT find a value of $a_{\rm cm} = 3.2$. Unfortunately, Hachisu (1986b) does not present numerical values for the separation at this turning point so we are unable to compare to this work. In contrast, although Rasio & Shapiro (1994) find a find an instability for the $\gamma = 3$ case of $a_{\rm cm} \approx 2.97$ based on hydrodynamic simulations, the equilibrium sequence they obtain on the basis of relaxation methods using their SPH code yields a turning point at $a_{\rm cm} \approx 2.7$. This result can be contrasted with the results of NT and our own results which show a turning point in both energy and angular momentum at substantially larger separations. However, the semi-analytic results presented in both LRS and in Lai et al. (1994b) show a turning point at $a_{\rm cm} = 2.99$. We will discuss the implications of this turning point for hydrodynamic evolution of a binary system in a subsequent section of this paper.

3.3. Initial Data for Hydrodynamic Models

Our numerical hydrodynamics method requires the density to be non-zero everywhere on the computational grid. Therefore, we include a low density (≈ 1 g/cm^3) "atmosphere" as a background in regions where stellar matter is not present. We have varied the density between (1 - 10^3g/cm^3) in our hydrodynamic simulations and have found that this has no discernible effect on the dynamics of the simulations. Other models (Ruffert et al. 1995, 1996) of neutron star mergers that have been carried out with Eulerian codes have had to employ much higher densities (10^9 g/cm^3) for the surrounding material.

However, adding matter in regions outside the stars presents two difficulties for hydrodynamic simulations. First, such matter will not in general be in hydrodynamic equilibrium if it has the same entropy as the matter in the stars. Thus at the beginning of the simulation it will immediately infall towards the stars and form an accretion shock at the surface of the stars. This accretion shock, while not physically troublesome because of the low density of the material in the atmosphere, will have the undesirable numerical effect of driving the timestep determined by the Courant stability condition to a very small timestep because of the high infall velocities. In order to counter this effect we make the atmosphere hot, i.e. we set the energy per baryon in the atmospheric material to approximately $35 - 40$ MeV. This has

the effect of preventing the atmosphere from falling down onto the neutron star surfaces. Furthermore we decrease this energy slightly with distance from the stars so as to achieve a configuration that is slightly more hydrostatically stable.

A second problem originates if the atmosphere is put in place with a non-zero velocity with respect to the stars. If the matter is placed on the grid with zero velocities in the lab frame, the motion of the stars quickly sweeps up the matter into a bow shock on the front sides of the orbiting stars. In a circumstance similar to the accretion shock mentioned in the previous paragraph, the bow shock has the numerical effect of driving the Courant timestep to zero. In order to avoid this problem, calculations that are carried out in the lab frame have an atmosphere with an initial velocity such that the material is rotating about the center of the grid at the same speed with which the stars are revolving. Near the edge of the grid the velocity of the atmospheric material is slowly tapered to zero so as to avoid a shock forming at the edge of the grid and to keep the velocity below the speed of light. In the case of rotating frame simulations the velocity of the atmospheric material is set equal to zero in the rotating frame.

These two steps obviate the problem of having the matter accrete onto the stars. We wish to note that this method requires no additional machinations to treat the material outside the stars; the evolution of the material is described by the numerical solution of the Euler equations.

4. Self-gravitating Hydrodynamics Numerical Methods

One of the more difficult aspects of self-gravitating hydrodynamics is the need to self-consistently solve both the partial differential equations describing the dynamics of the fluid and the equation(s) that describe the gravitational field arising from that matter. One of the origins of this difficulty in the Newtonian case is the different mathematical character of the two sets of equations: the Euler equations are hyperbolic while the Poisson equation is elliptic in nature. While the Euler equations can be numerically solved by explicit techniques, the Poisson equation requires an implicit solution. Since the two sets of equations are coupled by the gravitational acceleration term in the gas momentum equation, one must take care that the numerical methods employed for these coupled equations adequately maintain all the desirable properties of the total system such as angular momentum conservation and total energy conservation. In this section we compare several methods for these calculations that employ various methods for treating the gravitational acceleration term.

We wish to emphasize that no numerical scheme that solves the linear gas momentum equations in three dimensions will guarantee the numerical conservation of angular momentum. The converse is also true: if one solves the angular momentum equations in three dimensions the solution will not in general numerically satisfy the linear momentum equations. This discrepancy arises from the fact that the finite-differencing of the underlying partial differential equations reduces them to algebraic equations that must be solved for the new values of the density, internal energy, and velocity. Thus the five Euler equations are sufficient to algebraicly determine the five variables. The finite-differencing of the angular momentum conservation equations will be different from the linear momentum equations and thus give rise to five additional equations that must be algebraicly satisfied by the same five variables. The problem is algebraicly over-constrained. Despite the fact that the linear gas momentum and linear angular momentum equations can be easily shown to be equivalent, i.e. that conservation of linear momentum guarantees the conservation of angular momentum and vice versa, there is no such equivalence between the finite-difference analogs to these two vector equations. A similar statement can be made about the gas energy equation and

the total energy equation.

The issue of importance to simulations of orbiting neutron stars is how badly does numerical conservation break down over the course of a simulation? That is, how badly conserved are the angular momentum and total energy over the course of a simulation? We will consider these issues for several different schemes for coupling the Poisson and Euler equations. We will also show that a superior choice among these schemes emerges from these comparisons. This is vital for quantitatively accurate models of binary neutron star mergers where it is necessary to conserve both angular momentum and energy in order to ensure that orbital decay is physical and not the spurious result of numerical non-conservation.

4.1. Coupling gravity to the hydrodynamics

As we have previously mentioned in subsection 2.2, there are two possible orders of update of the source and transport portions of the Euler equations. These two possibilities are illustrated algorithmicly in Figure 2. The ZEUS algorithm of Stone & Norman employs the order of update shown on the left of Figure 2 while our V3D code employs the method shown on the right. As we have shown in Figure 3 there is no substantive difference between these two method in the case where self-gravity is not relevant. However, in the self-gravitating case these two approaches admit different possibilities for calculating the gravitational acceleration in the gas momentum equation.

In the case of the ZEUS algorithm, the solution of the source step first requires the gravitational potential in order to calculate the gravitational acceleration in the gas momentum equation. Hence the need for first solving the Poisson problem as described on the left side of Figure 2. Since the density at the new time (t^{n+1}) is not *a priori* known at the beginning timestep (at time t^n), the right hand side of the Poisson equation can only be constructed using the density that is known at time t^n. Thus the Newtonian gravitational potential is known only at time t^n. Consequently the gravitational acceleration term which is calculated from the Newtonian potential is not time-centered between times t^n and t^{n+1}.

Our code, V3D, performs the advection step before the source step that updates the Lagrangian terms (the terms on the right hand side of the hydrodynamics equations). This ordering, advection before the source update, allows the choice of computing the right-hand side of the Poisson equation, $4\pi G\rho$, with the density at the old time step (time lagged), the new time step (time advanced), or the average of the two densities (time centered). The finite-difference expressions for these choices are:

$$\begin{aligned}
\left(\nabla^2\Phi\right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= 4\pi G\rho^{n}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} && \text{time lagged}\\
\left(\nabla^2\Phi\right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= 4\pi G\rho^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} && \text{time advanced}\\
\left(\nabla^2\Phi\right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= 2\pi G\left(\rho^{n}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + \rho^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}\right) && \text{time centered}
\end{aligned}$$

This choice can play a significant role in the dynamics of a simulation. For example, in physical situations where the gravitational acceleration is always increasing in time, the use of the time-lagged centering will always underestimate the gravitational acceleration. Over long time scales this consistent underestimate can lead to significant deviations from the true physical behavior of the system. In the case of orbiting binary stars this could lead to non-physical evolution of the orbits. Finally, the choice of

time-centering for the gravitational acceleration can have a significant impact on the conservation of both angular momentum and total energy.

Both total energy and angular momentum conservation are vital in achieving good neutron star merger models. A lack of conservation of either of these two quantities could lead to unphysical inspirals and both qualitatively and quantitatively incorrect outcomes of the simulations. Therefore, it is necessary for us to examine how well both of these quantities are maintained as various choices are made for the gravitational acceleration coupling method.

4.2. Comparison of Gravitational Coupling Schemes in Rotating and Fixed Frame Calculations

In order to ascertain how the time centering affects the dynamics of binary orbits we have conducted a simple test. We have placed two spherical $\gamma = 2$, $1.4M_{\odot}$ polytropes in a circular orbit with a separation of $4R_{\star}$ where $R_{\star} \approx 9.55$ km is the radius of the spherical polytrope. At such wide separations the tidal distortion of the polytrope is minimal and the spherical approximation is valid. This latter point will be confirmed in a subsequent section of this paper where we compare spherical and equilibrium initial data.

In the physical situation described in the previous paragraph, the two stars should remain in perfectly circular orbits with constant angular momentum. For this reason we have carried out six simulations in which we compare the effects of the three time centerings for both the rotating frame and inertial frame cases. The results of these six simulations are shown in Figure 7, which depicts the trajectories of the centers of masses of the stars, and the barycenter of the system, in the orbital plane. Note that the trajectories are terminated at the point where the stars merge or where the simulation was stopped (if a merger did not occur). The comparison among the matrix of plots reveals that the choice of centering

Fig. 7.— The trajectories of the centers of masses of both stars and the binary system barycenter or center of mass (labeled as CM). The plots in the left column correspond to rotating frame calculations while the right column are inertial frame models. The top row of plots are time–advanced, the middle row are time-centered, while the bottom row is time-retarded. The total time of each simulation is indicated at the top of each panel.

has a major impact on the evolution of the orbits. The results of the ZEUS algorithm, which employs a time–lagged centering for the gravitational acceleration and is carried out in the inertial frame (depicted in the bottom right panel), show a completely spurious inspiral of the two stars in the first orbit. In contrast, the middle right and the top right panels show inertial frame models with time–centered and time–advanced gravitational acceleration couplings. While the decay of the orbit is diminished with the time–centered and time–advanced couplings, the overall evolution of the orbits is still unstable. In a forthcoming paper, (Calder et al. 2000) we shall show that this is a common feature of hydrodynamics simulations of this problem that employ inertial frames. The decay of the orbits in the inertial frame case is due to the non-conservation of angular momentum. This is shown directly in Figure 8 where the angular momentum evolution for inertial frame models in the time–centered and time–advanced cases are plotted over the first millisecond. We have carried out additional models for each of these cases with a series of decreasing Courant fractions. We define the Courant, or CFL, fraction as the ratio of our actual timestep

to the maximal possible hydrodynamic timestep as determined by the timestep control for the ZEUS/V3D algorithm (see SN for details). In most simulations we employ a CFL fraction of 0.4. The time-lagged models that we have carried out have revealed an even larger decrease in the angular momentum as a function of time than the time-centered models, which explains the rapid inspiral seen in the bottom right panel of Figure 7.

As Figure 9 shows, the lack of conservation of angular momentum is clearly related to the size of the timestep. A perfect algorithm would show no evolution of the angular momentum. Additionally, while the time–lagged case is much worse than the time–advanced case, both show a significant change in angular momentum over the first millisecond of evolution. In the time-lagged case this loss results in the decreasing orbits seen in Figure 7. In the time-advanced case the orbit outspirals and the system eventually

Fig. 8.— The evolution of the angular momentum about the z-axis in inertial frame models for time-centered (dashed lines) and time-advanced (solid lines) gravity couplings for a series of CFL fractions, $a_{\rm CFL}$. For the time centered couplings the CFL fractions were: (bottom to top) $a_{\rm CFL} = 0.06, 0.03, 0.015$, and 0.0075. For the time advanced couplings the CFL fractions were: (top to bottom) $0.4, 0.2, 0.12, 0.006, 0.003$, and 0.0015.

acquires a small drift due to the slight interactions with the boundaries. This drift becomes noticeable after many orbits. The artificial loss of angular momentum in the calculation is due to the inability of the finite-difference scheme to maintain conservation of both angular momentum and linear momentum. While this loss is mitigated through the use of time-advanced gravitational centering it is still sufficient to cause an unphysical inspiral of the system.

Fig. 9.— The change in the the total angular momentum (in units of 10^{49} erg seconds after 1 millisecond as a function of the CFL fraction $a_{\rm CFL}$ for the time-advanced (solid line) and time-centered (dot-dashed line) schemes.

The use of a rotating frame helps to minimize the effects of angular momentum loss. With a rotating frame it is possible to choose the angular velocity of the frame so that the motion of the stars with respect to the frame is minimized. The advantages of employing a rotating frame are clearly shown in Figure 7. In the rotating frame the advection of the stars across the grid is minimized and the angular momentum is conserved to a much higher degree. Nevertheless, the time centering of the gravitational acceleration plays a role in determining the dynamics of the orbits. The best combination of techniques is illustrated in the top left panel which shows the results from a simulation using both a rotating frame and the time-advanced coupling. This particular scheme maintains stable orbits for the two stars for more than seven orbits at which point the simulation was terminated. The simulation has shown no significant change in the orbits of the two stars over the course of the simulation. An examination of the angular momentum evolution for this simulation, in Figure 10, shows that the total angular momentum is well conserved. The lines in this figure show the angular momentum contained in matter above various density thresholds. Note that nearly half of the angular momentum is contained in the high-density cores of the polytropes. Also note that the $\rho > 2.5 \times 10^{14}$ g/cm^3 line shows that that there is initially a slight re-adjustment in the angular momentum distribution as the star relaxes on the grid. Nevertheless the total angular momentum is fairly

well conserved over the course of the simulation.

In contrast the time-lagged inertial-frame case, shown in Figure 11, reveals poor angular momentum conservation. This simulation shows a steady decline in the angular momentum at all densities. In particular the high density core has lost most of the angular momentum. The loss of angular momentum terminates approximately when the two stars have coalesced into a single central object. At this point the object is fairly axisymmetric and could almost be though of as having achieved a steady state. Under these circumstances the time centering of the gravitational acceleration is not as critical as it was prior to coalescence and consequently the angular momentum is better conserved at late times.

The behavior of the total energy (Figure 12) in the the time-lagged inertial-frame case shows a slight decline which is not nearly so dramatic as the behavior of the angular momentum. Again, the decline ceases after coalescence. Note that Figure 12 clearly shows the transfer of gravitational potential energy to kinetic energy during the inspiral and coalescence. The total internal energy changes very little throughout the coalescence. The time-advanced rotating frame case (Figure 13) shows a very steady behavior for all of the energies, with no substantial change throughout the length of the simulation.

Finally, we wish to point out that virtually none of the loss of angular momentum or energy is due to dissipation by the artificial viscosity terms in the gas energy and gas momentum equations. The total dissipation due to these terms is tracked throughout the simulation, and it is many orders of magnitude below the other energy and angular momentum scales involved. This includes the case where the stars have coalesced. We find no significant amount of shock generated dissipation as the stars merge in any of our models.

Fig. 10.— The evolution of the angular momentum for the time-advanced rotating-frame case. The lines show the total angular momentum contained in matter above the listed density. The line labeled “grid” indicates the total angular momentum on the entire computational grid.

Fig. 11.— The same as 10 except for the time-lagged inertial-frame case.

Fig. 12.— The evolution of the internal, potential, kinetic, and total energies for the time-lagged fixed-frame case.

We have assumed that the angular velocity of the rotating frame with respect to the inertial lab frame is a constant. Thus in situations where the two stars inspiral due to physical processes, the stars will acquire a non-zero velocity with respect to the rotating frame. In this situation one might suspect that the angular momentum conservation might begin to break down as the stars begin moving with respect to the grid. However, in the next section we shall show that the angular momentum conservation is still well–maintained even in the case where the stars inspiral and merge.

Fig. 13.— The same as 12 except for the time-advanced rotating-frame case.

The significant amount of energy and angular momentum non-conservation in the fixed frame calculations clearly establish that there are significant problems associated with their use in modeling binary neutron stars. We are currently surveying other hydrodynamic methods to see if the same difficulties are present in these other schemes. In order to avoid the problems associated with the inertial Cartesian frames we have chosen to employ the rotating frame, time-advanced gravitational acceleration scheme as our preferred method for simulating orbiting and inspiral binary neutron stars. Using this method we turn to the study of the stability of equilibrium models.

5. Dynamical Studies of Newtonian Models

5.1. Stability of Equilibrium Equilibrium Models

It has been known for some time that even in the purely Newtonian case that tidal instabilities can drive coalescence in binary polytropic systems. Recent semi-analytic stability analyses have been performed by Lai, Rasio, and Shapiro (1993a,1993c,1994a,1994c) and Lai and Shapiro (1995). These models, which treat the binary polytropes as self-similar ellipsoidal figures of equilibrium, have found that close polytropic binary systems may be unstable to both dynamic and secular instabilities. In this context we refer to a dynamical instability as one that takes place on the orbital timescale of the binary system while secular instabilities involve dissipative processes that may occur on much longer timescales. The presence of these instabilities was confirmed numerically by Rasio and Shapiro (1992,1994,1995) using SPH hydrodynamics methods. More recently, New and Tohline (1997) have performed similar calculations using Eulerian hydrodynamics methods and have found results for the $\gamma = 2$ ($n = 1$) polytropic sequences that differ from those of Rasio & Shapiro. In this section we discuss our investigations of these equilibrium sequences using the time-advanced rotating frame hydrodynamic scheme discussed in the previous section.

The initial data for these equilibrium sequences was constructed as described in section 3. The models that we will discuss were all run at $129 \times 129 \times 129$ resolution with an approximate size of 65 km in each dimension. The grid used to construct the equilibrium data was the same grid that was used for the hydrodynamic simulation, thus obviating any introduction of error by remapping the data onto a new grid. Since our primary interest is in neutron star mergers we have only carried out simulations for $\gamma = 2$ and $\gamma = 3$ equilibrium sequences.

The results of our simulations for the $\gamma = 2$ equilibrium sequence summarized in Figure 14, which shows the time evolution of the separation between the centers of mass between the two stars. We have utilized the center-of-mass of the stars to define their separation in the same fashion as LRS. In contrast with NT we have found pressure maxima to be ill suited for use as a separation diagnostic since extremely small changes in the values of the pressure in a given zone as the stars move can cause a discrete jump in the location of the maxima. Because the center-of-mass is density-weighted, the location of these points changes smoothly. Note that the binary systems with initial separations greater than approximately 28 km seem to be stable over may orbits while those with initial separations less than this radius do not. Normalized to the value of the unperturbed polytropic radius this cutoff corresponds to a separation of $a_{\rm cm} \approx 2.8$.

Fig. 14.— The evolution of the center-of-mass separation, $a_{\rm cm}$ (in kilometers), for $\gamma = 2$ equilibrium binary systems

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This is in close agreement with the minimal energy and angular momentum separation which was found for this $\gamma = 2$ sequence (shown in Figure 5). This also corresponds to the point at which the equilibrium sequence transitions from detached to connected binary systems. This result is in close agreement with the predictions of LRS. However, while we agree with the conclusion of Rasio & Shapiro (1992) that models with $a_{\rm cm} = 2.8$ are unstable, we do not agree with their finding that the models inspiral on a timescale of 1-1.5 times the initial orbital period. We find that the inspiral occurs on timescales of $3 - 5$ times the orbital period. This timescale for evolution of these close systems seems to closely follow those of NT. While the simulations of NT were not carried out for a time sufficient to show instabilities, the closest separation systems of NT showed an outward evolution comparable to ours over the first four orbital periods.

While we seem to agree with the numerical results of NT, we disagree with the conclusion drawn by NT regarding stability of the $\gamma = 2$ polytropic sequence. We see all systems interior to the minimum energy separation inspiral on the timescales of several, i.e. 3-5, orbits. The hydrodynamic simulations of NT have stopped at four orbital periods. However, most of our coalescing systems inspiral at precisely that time. There is no reason to believe that the dynamical timescale for the inspiral must only be 1-2 orbital periods. While the inspirals could be a result of a secular instability triggered by numerical inaccuracies within the code, it seems more likely that the dynamical process may take slightly longer than what is anticipated by NT.

An interesting feature emerges from Figure 14 where we note that the systems with the smallest separations spiral out slightly towards the minimum energy point before undergoing tidal disruption. Similar behavior was seen by NT, who unfortunately terminated their calculations before the point where we see the inspiral occur. This can be seen from Figure 12 of NT, which shows the growth in the moment of inertia of their closest system. This evolution can be interpreted as an instability that is driving the system towards a lower energy configuration at separations of $a_{\rm cm} \approx 2.85$.

The quality of the total energy and angular momentum conservation for the coalescing models is paramount. As the stars coalesce a significant amount of matter is rapidly advected about the grid even in the rotating frame calculations. One might suspect that the quality of angular momentum and energy conservation might break down under such circumstances. However we have found that this does not seem to happen. This is illustrated by the results for the $a_{\rm cm} = 2.78$ model which is typical of the coalescing cases. As Figures 15 and 16 indicate, both the angular momentum and the energy are well conserved. The coalescence begins at a time of approximately 8 msec at which time there is a substantial transfer of angular momentum from the high density material to lower density material. As a result of this angular momentum transfer and the disruption of the stars tidal "arms" are formed of material that is stripped from the stars. These tidal arms contain a significant fraction of the total angular momentum. Some of the material in these arms is swept off of the grid, carrying with it angular momentum. In Figure 15 we separately track the total angular momentum on the grid at every instant in time along with the cumulative total of the angular momentum swept off of the grid. The total angular momentum is the sum of these two curves. The components of the angular momentum displayed in Figure 15 are entirely composed of angular momentum about the z-axis; the x and y components are effectively zero. The angular momentum on the

Fig. 15.— The evolution of the angular momentum in the $a_{\rm cm} = 2.78$ run. The dashed line indicates the total angular momentum in the domain of integration as a function of time. The dot-dashed line indicates the loss of angular momentum from the grid obtained by integrating the angular momentum flux over the boundary of the domain and over time. The solid line indicates the sum of the angular momentum in the domain plus the lost angular momentum.

grid undergoes a sharp decline during the merger as matter flows off the grid. This is also reflected in the rise of the cumulative total angular momentum that has been advected off the grid by the matter. Yet the total angular momentum remains quite well conserved. We wish to emphasize that we have not adjusted the rotation speed of the frame as the stars have inspiraled; the grid has maintained a constant rotation speed with respect to the laboratory inertial frame. The introduction of a time-varying grid rotation speed would complicate the hydrodynamic equations and would at best yield only relatively small improvements in angular momentum conservation. The evolution of the three components of the total energy is shown in

Fig. 16.— The evolution of the potential, kinetic, internal, and total energies for the $a_{\rm cm} = 2.78$ run

16. There is a slight rise of a few percent in the total energy over the course of the entire simulation but no significant jump during the coalescence. A modest transfer of kinetic and potential energy occurs during the the merger but this does not have a pronounced effect on the conservation of total energy.

The hydrodynamic evolution of models from the $\gamma = 3$ equilibrium sequence shows an instability at a separation of approximately $a_{\rm cm} \approx 2.85$ in good agreement with the minimum energy and angular momentum separation of $a_{\rm cm} = 3.0$. The evolution of three models with $129 \times 129 \times 129$ resolution is shown in Figure 17 where the binary center-of-mass separation is shown. Because of the difficulty of

Fig. 17.— The evolution of the center-of-mass separation, $a_{\rm cm}$, for $\gamma = 3$ equilibrium binary systems
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constructing high-resolution equilibrium models for the $\gamma = 3$ sequence, we have carried out only five simulations bracketing the predicted point of instability. Our results again agree with the location of the instability identified in RS94 (see RS94 Figure 3). RS94 found the instability occured at $a_{\rm cm} = 2.97$, a value within 10% of the semi-analytic prediction of LRS of a point of instability of $a_{\rm cm} = 2.7$. In contrast, we do not agree with the results of NT who find that binaries at larger separations, e.g. $a_{\rm cm} = 3.1$ models, are unstable to merger (see Figure 13 of NT). Our models with this initial separation exhibit no sign of instability. A puzzling fact about the NT results for the $\gamma = 3$ sequence is that even the largest separation model with an initial value of $a_{\rm cm} = 3.41$ show signs of a slow orbital decay. Neither we, nor RS, see such behavior.

5.2. Comparison of Models Using Equilibrium and Non-equilibrium Data

Many numerical investigations of the dynamics of binary neutron star coalescence have employed spherical stars as initial data. As we discussed in section 4.2 the tidal distortions for widely separated stars are small and one can often assume that spherical stars are a good approximation to the true equilibrium fluid bodies. This assumption clearly breaks down as the separation between the two stars is reduced. The critical question is as what point does this break down occur?

In order to clarify the realm of validity of the spherical initial data approximation we have carried out a series of simulations using $\gamma = 2$ and $\gamma = 3$ polytropes as initial data. The initial separations were varied in the same fashion as the the series of runs for equilibrium data models. The evolution of these separations for the $\gamma = 2$ case is shown in Figure 18. At larger separations the systems are stable for long

Fig. 18.— The evolution of $a_{\rm cm}$ for $\gamma = 2$ spherical-star binary systems
.

timescales; the separations to do not significantly evolve. The small oscillations present reflect the fact that the spherical stars are not in perfect equilibrium initially and consequently the evolving systems undergo small epicyclic oscillations. Similar behavior has been seen by RS94. For binary separations nearer the equilibrium sequence stability limit we can see substantial differences between the $\gamma = 2$ binaries shown in Figure 18 and their equilibrium counterparts shown in Figure 14. For systems with separations less than 30 km, the orbital separation is diminishing. In the equilibrium case these systems are stable as is seen in Figure 14. Similar behavior is seen in the $\gamma = 3$ case as is shown in Figure 19, which compares the equilibrium and spherical-star models. The rate at which systems inspiral is clearly high for systems

Fig. 19.— A comparison of the evolution of $a_{\rm cm}$ for $\gamma = 3$ equilibrium and spherical-star binary systems
.

of smaller initial separation. In fact, the closest systems are disrupted almost immediately. However, the systems with initially wider separations show no sign of instability.

The results clearly indicate that for systems with initial separations well beyond the tidal instability limit that the spherical-star approximation is quite acceptable. This point is very important for the case of the coalescence of two rotating neutron stars where one would have to solve the compressible Darwin–Riemann problem in order to obtain equilibrium initial data. By starting sufficiently far beyond the tidal instability limit one may be able to effectively employ static models of isolated rotating neutron stars as initial data for binary configurations. Furthermore, the isolated star approximation, using post-Newtonian models for the isolated stars, could also greatly simplify the construction of initial data for post-Newtonian simulations as well.

6. Conclusions

Self-gravitating hydrodynamic models for binary neutron star phenomena pose some unique challenges for numerical modelers. Since the use of Eulerian hydrodynamics techniques is prevalent in Newtonian, post-Newtonian, and relativistic models of binary neutron star coalescence, it is vital that we have a good understanding of the role that the numerical techniques play in determining the outcome of the models. To this end, we have carried out a number of studies designed to compare rotating and inertial frame Newtonian hydrodynamic models as well as to compare several choices that could be made for the coupling of gravity and matter. The lessons that are learned from this efforts will be invaluable for post-Newtonian and relativistic models as well.

We have been able to show that a combination of a rotating frame of reference and a time-advanced gravitational acceleration centering in the gas momentum equation yields adequate angular momentum conservation for the orbiting and merging binary problems. In contrast, we have found that the use of a inertial laboratory frame together with a time-lagged gravitational coupling yields incorrect results. The inertial frame methods produces a substantial angular momentum loss that leads to a spurious inspiral of what should be a stable Newtonian binary system. This result indicates that the use of inertial frames which involve stars advecting across the grid should be avoided where possible.

We have found a reliable method for constructing equilibrium initial data on the hydrodynamic grids for use in hydrodynamic simulations. This method employs a method of solving the Poisson problem, including the determination of consistent boundary conditions, for isolated self-gravitating systems without having to resort to multi-pole expansions of the mass distribution. This iterative method is easily implemented and produces initial data that is consistent with the hydrodynamic grid. Using this method we have constructed equilibrium sequences that closely agree with the semi-analytic calculations of Lai, Rasio, and Shapiro for $\gamma = 2$ and $\gamma = 3$ polytropic sequences. While we see qualitative similarities with the results of New & Tohline the locations of the minima differ somewhat from theirs and are closer to the LRS predictions.

Using the initial data from our equilibrium sequences we have investigated the stability of these models. Our results are in very close agreement with the numerical SPH models of Rasio and Shapiro. In contrast, we find that we disagree with the conclusions of New & Tohline on the stability of the $\gamma = 2$ and $\gamma = 3$ equilibrium sequences.

Finally, we investigate the effects of using the isolated star approximation for initial data. We find that for separations modestly greater than the tidal instability limit that the use of isolated polytropes for initial data has little influence on the subsequent evolution of the binary system. This point justifies the use of the isolated star approximation for the construction of equilibrium data for rotating, post-Newtonian, and other complex binary neutron star systems.

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A. Hydrodynamic Algorithm

In this appendix we briefly describe the details of our hydrodynamic algorithm. The finite-differencing methodology is identical to that for the ZEUS algorithm as described by Stone & Norman (1992). An exception is the addition of the Coriolis and centrifugal force terms, which do not appear in SN. Additionally, we have employed only Cartesian coordinates, which significantly simplifies the equations as compared to the generalized coordinates of SN. For intimate details of the algorithm we refer the reader to SN.

We wish to point out one important difference of our algorithm (V3D) from the ZEUS algorithm: although the finite-difference stencils of the equations are identical, the order of updates differs significantly as discussed in section 2.2. Nevertheless, we still employ a multi-step (operator split) methodology as discussed in section 2. In the SN nomenclature the Euler equations are broken up into the *transport* and *source* terms. The transport step results in the update of the hydrodynamic variables as due to the advective terms of the Euler equations, while the source step results in the updates due to the remaining terms. We describe each of these in turn.

A.1. Transport Step

In the transport step, equations (11)-(13) are updated. These equations represent only the advective part of the hydrodynamic evolution. Because of the 3-dimensional nature of these equations we employ the widely used dimensional operator splitting technique of Strang (1968) which decomposes the 3-D update into a series of 1-D updates in each dimension. For simplicity we will describe our updates in only the x-direction. The application to the y- and z-directions is obvious.

In each dimension equations (11)-(13) are simple conservation laws of the form

$$\frac{\partial q}{\partial t} + \frac{\partial F(q)}{\partial x} = 0 \tag{A1}$$

where q is generic variable representing and advected quantity and $F(q)$ is the flux of that quantity in the x-direction. For the five equations (11)-(13) q takes the respective forms of ρ, E, or ρv_i, while $F(q)$ takes the forms of ρv_x, $E v_x$, and $\rho v_i v_x$. The fluxes are calculated for the x-faces of a cell centered around the point at which q is defined. Note that these cells will differ for the five variables with the exception of ρ and E which are both defined at the same point. For a given cell the update of q will take the form

$$\tilde{q} = q^n + \Delta t(F_l(q) - F_r(q))/\Delta x \tag{A2}$$

where F_r and F_l are the fluxes on the right and left x-faces of the cell, ΔX is the cell width, and q^n is the known value of q at timestep t^n. The notation $\tilde{q}$ is used to denote that this is only a partial update of q only due to advection. The advection scheme utilizes the consistent advection scheme of Norman (1980) which ties the energy and momentum fluxes to the mass flux, i.e. we define

$$F(E) = \varepsilon^\star F(\rho), \tag{A3}$$

where ε is the internal energy per gram and

$$F(\rho v_i) = v_i^\star F(\rho), \tag{A4}$$

where

$$F(\rho) = (\rho v_x)^\star \tag{A5}$$

where the $\star$ indicates that values of variables are calculated using the monotonic advection scheme of van Leer (1977). The implementation of this scheme is detailed in SN and we refer the reader to that paper for more information.

A.2. Source Step

The implementation of the source step in nearly identical to that of SN. In this step we solve equations (14) and (15). Note that there are no source terms for the continuity equation and thus the source step does not affect any change in the density. Thus the updates of the internal energy density, E, and the momentum density components, ρv_i, are of the form

$$q^{n+1} = \tilde{q} + \Delta t(\text{source terms}). \tag{A6}$$

This update makes use of the intermediate result obtained from the previously undertaken transport step. Since the details of the finite-differencing for most of these source terms are given in SN we refer the reader to that work for further information. However, in the gas momentum equation the the Coriolis and centrifugal force terms are differenced as

$$\begin{aligned}(v_x)^{n+1}_{i,j+1/2,k+1/2} &= (\tilde{v}_x)_{i,j+1/2,k+1/2} + \omega^2 \Delta t(x_i - x_c) + \frac{\omega \Delta t}{2}\Big[\\ &\quad (v_y)^n_{i+1/2,j+1,k+1/2} + (v_y)^n_{i+1/2,j,k+1/2} \\ &\quad + (v_y)^n_{i-1/2,j+1,k+1/2} + (v_y)^n_{i-1/2,j,k+1/2}\Big] \qquad \text{(A7)} \\ & \qquad \text{(A8)}\end{aligned}$$

where x_c is the x-coordinate of the grid center. In equation (A8) we have employed the difference notation as detailed in SN. The update for the y-velocity component is similarly given by

$$\begin{aligned}(v_y)^{n+1}_{i+1/2,j,k+1/2} &= (\tilde{v}_y)_{i+1/2,j,k+1/2} + \omega^2 \Delta t(y_j - y_c) - \frac{\omega \Delta t}{2}\Big[\\ &\quad (v_x)^n_{i+1,j+1/2,k+1/2} + (v_x)^n_{i+1/2,j-1/2,k+1/2} \\ &\quad + (v_x)^n_{i,j+1/2,k+1/2} + (v_x)^n_{i,j-1/2,k+1/2}\Big] \qquad \text{(A9)}\end{aligned}$$

where y_c is the y-coordinate of the center of the grid. Note that since we are considering non-inertial frames that rotate around the z-axis, there are no Coriolis or centrifugal contributions to the z-component of the momenta. The Coriolis and centrifugal updates are completed in the middle of the source step. After the updates to the momenta due to pressure and gravitational accelerations have been completed, the velocities are calculated from the momenta. Equations (A8) and (A9) are then applied to get the non-inertial force updates to the velocities. Finally, the viscous stress updates to the velocities are completed. As the last step the source terms for the gas energy equation are solved in order to obtain the new internal energy.

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